Abbes Laghrour University of Khenchela
Faculty of sciences and technology
Department of Civil Engineering

Numerical method
Duration:1h:30
20/05/2025

Exam

Exercise 01

Consider the function **F(x)** defined by: $F(x)=2x\cos(2x)-(x+1)^2$, where **x** is given in **radians**.

1. Can the Bisection method be applied to calculate α_1 on the interval $I=[-3;-2]$? Justify.
2. Let $\varepsilon=10^{-4}$, determine the number of iterations required to approximate, using the Bisection method, the root α_1 located in **I**.
3. Given that $a_0=-3$ and $b_0=-2$, calculate the first five iterations using this method.

Exercise 02

Consider the function f defined by:

x_i	-1	0	1	2
f_i	-1	1	0	0

1. What is the **degree** of the polynomial represented by these points.
2. Construct the **Newton interpolation polynomial** passing through these points.

Exercise 03

Given the integral: $I=\int_0^1 e^{-x^2}dx$

1. Approximate the integral I using:

   -The composite midpoint rule with **n=4** subintervals.

   -Trapezoidal rule

2. Evaluate the error for each method.
3. Calculate **the exact value** of the integral.

Exercise 04

Consider the following Cauchy problem:

$$\begin{cases} y'(t)=y(t)-\dfrac{2t}{y} \\ y(0)=1 \end{cases}$$

1. Calculate the approximate solution of this equation at **t=0.4** using the second order Runge-Kutta method with step size **h=0.1**.

Abbes Laghrour University of Khenchela
Faculty of sciences and technology
Department of Civil Engineering

Numerical method
Duration:1h:30
20/05/2025

Corrected type

Exercise 01: (05) pts

$$F(x) = 2x\cos(2x) - (x+1)^2$$

1. This function is well defined and continuous, so we can be applied Bisection method 0.25

1.1.justification

F (-3) =−9.76102 0.25
F (-2) = 1.61456 0.25 $\Longrightarrow$ **F (-3) *F (-2) <0** 0.25 0.25

So, according to the intermediate value theorem, there exists at least one root $\alpha \in]-2,-3[$ such that **f(α)=0.** 0.25

Furthermore, 0.25 0.25

F'(x)=2cos(2x)-4xsin(2x)-2(x+1), ∀x∈ [-2, -3] F'(x)>0 the function is increasing (F is monotone).

We deduce that the root of F is unique 0.25

2. Determine the number of iterations required

n≥((log(b-a)/ε)/log2)-1 0.25

n≥ ((log(-2-(-3))/10⁻⁴)/log2)-1 0.25

n≥ (log (1/0,0001)-log (2))/log2=12,28

n=13 0.25

3. Calculate the first five iterations

$x_0=(a_0+b_0)/2= (-2-3)/2=-(5/2)$. 0.25

	i	a_i	x_i	b_i	$f(a_i)$	$f(x_i)$	$f(b_i)$
0.25	0	-2	-5/2	-3	+	-	+
0.25	1	-2	-2,25	-5/2	+	-	-
0.25	2	-2	-2,125	-2,25	+	+	-
0.25	3	-2,125	-2,1875	-2,25	+	+	-
0.25	4	-2,1875	-2,2187	-2,25	+	-	-

0.25 5 -2,1875 -2,2031 -2,2187

Abbes Laghrour University of Khenchela

Faculty of sciences and technology

Department of Civil Engineering

Numerical method

Duration:1h:30

20/05/2025

Exercise 02 : (05 pts)

1. The degree n of the interpolation polynomial **P**?

Since the number of interpolation points is **4**, we have: 0,25 0,25

N=number of interpolation points-1=4-1=3 0,25

2. The third-degree interpolation polynomial that passes through the four points **(x_0,y_0), (x_1,y_1),(x_2,y_2),(x_3,y_3)** is the form:

$$P_3(x)=ax^3+bx^2+cx+d$$ 0,25

We observe that the given points are equidistant with an interpolation step of ***h* =1** is given by :

$$P_3(x)=f(x_0)+\frac{\Delta f(x_0)}{1!h}(x-x_0)+\frac{\Delta^2 f(x_0)}{2!h^2}(x-x_0)(x-x_1)+\frac{\Delta^3 f(x_0)}{2!h^3}(x-x_0)(x-x_1)(x-x_2)$$ 0,5

Where:

$$h=x_1-x_0=x_2-x_1=x_3-x_2=1$$ 0,25

The finite differences are calculated as follows:

x_i	$f(x_i)$	$\Delta f(x_i)$	$\Delta^2 f(x_i)$	$\Delta^3 f(x_i)$
$x_0=-1$ 0,25	$f(x_0)=-1$	$\Delta f(x_0)=1-(-1)=1+1=2$ 0,25	$\Delta^2 f(x_0)=-1-2=-3$ 0,25	$\Delta^3 f(x_0)=1-(-3)=4$ 0,25
$x_1=0$ 0,25	$f(x_1)=1$	$\Delta f(x_1)=0-1=-1$ 0,25	$\Delta^2 f(x_1)=0-(-1)=1$ 0,25	
$x_2=1$ 0,25	$f(x_2)=0$	$\Delta f(x_2)=0-0=0$ 0,25		
$x_3=2$ 0,25	$f(x_3)=0$			

Hence

$$P_3(x)=f(x_0)+\frac{\Delta f(x_0)}{1!h}(x-x_0)+\frac{\Delta^2 f(x_0)}{2!h^2}(x-x_0)(x-x_1)+\frac{\Delta^3 f(x_0)}{2!h^3}(x-x_0)(x-x_1)(x-x_2)$$

$$P_3(x)=-1+\frac{2}{1}(x+1)+\frac{-3}{2}(x+1)(x-0)+\frac{4}{6}(x+1)(x-0)(x-1)$$

$$=-1+2x+2-\frac{3}{2}x^2-\frac{3}{2}x+(\frac{2}{3}x^2+\frac{2}{3}x)(x-1)$$

$$=1+2x-\frac{3}{2}x^2-\frac{3}{2}x+\frac{2}{3}x^3+\frac{2}{3}x^2-\frac{2}{3}x^2-\frac{2}{3}x$$

$$P_3(x)=\frac{2}{3}x^3-\frac{3}{2}x^2-\frac{1}{6}x+1$$ N

This is the polynomial obtained using finite differences. $\boxed{2.0}$

2

Abbes Laghrour University of Khenchela

Faculty of sciences and technology

Department of Civil Engineering

Numerical method

Duration:1h:30

20/05/2025

Exercise 03:

Given the integral: $I = \int_0^1 e^{-x^2} dx$

Let $f(x) = e^{-x^2}$. The interval of integration is $[a, b] = [0, 1]$. 0,25

We are given $n = 4$ subintervals.
The step size $h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$. 0,25

The points for calculation are:

$x_0 = 0.0$
$x_1 = 0.25$
$x_2 = 0.50$
$x_3 = 0.75$
$x_4 = 1.00$

1. Approximate the integral I using:

a) The composite Midpoint Rule with $n = 4$ subintervals:

The formula for the composite Midpoint Rule is:

$I_M = h \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right)$ 0,5

$I_M = 0.25 \times [f(0.125) + f(0.375) + f(0.625) + f(0.875)]$

$I_M = 0.25 \times [0.984496 + 0.868700 + 0.676676 + 0.465134]$

$I_M = 0.25 \times 2.995006$

$I_M \approx 0.748752$ 0,5

The corresponding function values are:

$f(x_0) = e^{-0^2} = 1.000000$
$f(x_1) = e^{-0.25^2} = e^{-0.0625} \approx 0.939413$
$f(x_2) = e^{-0.50^2} = e^{-0.25} \approx 0.778801$
$f(x_3) = e^{-0.75^2} = e^{-0.5625} \approx 0.570221$
$f(x_4) = e^{-1.00^2} = e^{-1} \approx 0.367879$

For the Midpoint Rule, we need the midpoints:

$m_0 = (0.0 + 0.25)/2 = 0.125$
$m_1 = (0.25 + 0.50)/2 = 0.375$
$m_2 = (0.50 + 0.75)/2 = 0.625$
$m_3 = (0.75 + 1.00)/2 = 0.875$ 0,25

The function values at midpoints are:

$f(m_0) = e^{-0.125^2} = e^{-0.015625} \approx 0.984496$
$f(m_1) = e^{-0.375^2} = e^{-0.140625} \approx 0.868700$
$f(m_2) = e^{-0.625^2} = e^{-0.390625} \approx 0.676676$
$f(m_3) = e^{-0.875^2} = e^{-0.765625} \approx 0.465134$ 0,25

Abbes Laghrour University of Khenchela
Faculty of sciences and technology
Department of Civil Engineering

Numerical method
Duration:1h:30
20/05/2025

b) Trapezoidal Rule (Composite, with $n = 4$ subintervals):

The formula for the composite Trapezoidal Rule is:

$I_T = \frac{h}{2}[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)]$ 0,5

$I_T = \frac{0.25}{2}[f(0) + 2f(0.25) + 2f(0.50) + 2f(0.75) + f(1.00)]$

$I_T = 0.125 \times [1.000000 + 2(0.939413) + 2(0.778801) + 2(0.570221) + 0.367879]$

$I_T = 0.125 \times [1.000000 + 1.878826 + 1.557602 + 1.140442 + 0.367879]$

$I_T = 0.125 \times 5.944749$ 0,5

$I_T \approx 0.743094$

b- Midpoint Rule Error:

$$E_{PM}(f) = I(f) - I_{PM}(f) = f''(c)\int_a^b (x - \frac{a+b}{2})dx = \frac{f''(c)}{24}(b-a)^3$$

$$|E_{PM}(f)| \leq \frac{(b-a)^3}{24}M \quad \text{and} \quad |E_{PM_c}(f)| \leq \frac{(b-a)^3}{24n^2}M$$

where:

$$M = \max_{c\in[a,b]} |f''(c)|$$ 0,5

$E_{PMc}(f) \leq (1-0)^3/24(4)^2.M$

$f'(x)=-2xe^{-x2}$ $f''(x)=(4x^2-2)\, e^{-x2}$

On [0,1], M=f''(1) =0,73576

$E_{PMc}(f) \leq (1-0)^3/24(4)^2.0.73576$ 0,5

$E_{PMc}(f) \leq (1/384).0.73576=0.0019$

L'erreur de la méthode des Trapèzes est : $|E_{T_c}(f)| \leq \frac{(b-a)^3}{12n^2}M$ où $M = \underset{c\in[a,b]}{Max}|f''(c)|$ 0,5

On a :

$f(x) = e^{-x^2}$, $f'(x) = -2xe^{-x^2}$, $f''(x) = (4x^2-2)e^{-x^2}$

$|E_{Tc}(f)| \leq ((1-0)^2/12*4^2)*0,73576$

$|E_{Tc}(f)| \leq (1/192)*0,73576$ 0,5

$|E_{Tc}(f)| \leq 0,00383$

Exercise 04: (5 pts)

- Calculate the approximate solution of the differential equation at $t=0,4$ using the Runge-Kutta method of order 2 (RK2):

Given:

$$f(x,y) = y - \frac{2t}{y} \quad , \quad y(0) = 1.$$

- The interval is $[0, 0,4]$ with $h = 0,1$.
- The RK 2 method is given by:

$$y_{i+1} = y_i + \frac{1}{2}(k_1 + k_2)$$

Where:

$$k_1 = h f(t_i, y_i)$$

$$k_2 = h f(t_i + h, y_i + k_i)$$

- Applying the method step by step:

For $i = 0$ $\quad t_0 = 0$ 0,25

$k_1 = h f(t_0, y_0)$ 0,25

$= h f(t_0 + h, y_0 + k_0)$ 0,25

$= 0,1 \left(1 - \frac{2(0)}{1}\right) = \boxed{0,1}$

0,1118 1

$0,1 \left(1,1 - \frac{2(0,1)}{1,1}\right) = \boxed{0,0918}$

$y_0 + \frac{1}{2}(k_1 + k_2)$ 0,25

$+ \frac{1}{2}(0,1 + 0,0918) = \boxed{1,095}$

$t_1 = 0,1$ 0,25

For $i = 1$

$= h f(t_1, y_1)$, $y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$

$= h f(t_1 + h, y_1 + k_1)$

$0,1 \left(y_1 - \frac{2t_1}{y_1}\right) = 0,1 \left(1,095 - \frac{2(0,1)}{1,095}\right)$

$= 0,09126$ 0,25

$k_2 = 0,1 \left(1,095 + 0,0912 - \frac{2(0,1+0,1)}{(1,095 + 0,091}\right.$

$= 0,0848$ 0,25

$y_2 = 1,095 + \frac{1}{2}(0,0912 + 0,0848)$

$= 1,183$ 0,25

$t_2 = 0,2$ 0,25

For $i = 2$

$k_1 = h f(t_2, y_2)$, $k_2 = h f(t_2 + h, y_2 +$

$y_3 = y_2 + \frac{1}{2}(k_1 + k_2)$ 0,25

$k_1 = 0,1 \left(1,183 - \frac{2(0,2)}{1,183}\right) = 0,084$

$k_2 = 0,1 \left(1,183 + 0,084 - \frac{2(0,2+0,1)}{1,183 + 0,084}\right.$

$= 0,0793411$ 0,25

$y_3 = 1,183 + \frac{1}{2}(0,084 + 0,0793$

$= \cancel{1,1622}$ $1,2646$ 0,25

5

Scanné avec CamScanner

For $i = 3$ $\quad t_3 = 0{,}3$ (0,25)

$$k_1 = h f(t_3, y_3) \text{ (0,25)}, \quad k_2 = h \cdot f(t_3 + h, y_3 + k_1) \text{ (0,25)}$$

$$y_4 = y_3 + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = 0{,}1\left(1.2646 - \frac{2(0{,}3)}{1{,}2646}\right) = 0{,}10273$$

$$k_2 = 0{,}1\left(1.2646 + 0{,}10273 - \frac{2(0{,}3 + 0{,}1)}{1.2646 + 0{,}10273}\right) = 0{,}0782$$

$$y_4 = 1{,}2646 + \frac{1}{2}(0{,}10273 + 0{,}0782) = 1{,}3556 \text{ (0,25)}$$

For $i = 4$ $\quad t_4 = 0{,}4$ (0,25)

$$k_1 = h f(t_4, y_4), \quad k_2 = h \cdot f(t_4 + h, y_4 + k_1).$$

$$y_5 = y_4 + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = 0{,}1\left(1{,}3550 - \frac{2(0{,}4)}{1{,}3550}\right) = 0{,}0764 \text{ (0,25)}$$

$$k_2 = 0{,}1\left(1{,}3550 + 0{,}0764 - \frac{2(0{,}4 + 0{,}1)}{1{,}3550 + 0{,}0764}\right) = 0{,}0732$$

$$y_5 = 1{,}3550 + \frac{1}{2}(0{,}0764 + 0{,}0732) = 1{,}4298 \text{ (0,25)}$$

The approximate solution of the equation at $t = 0{,}4$ is

$$y(0{,}4) \simeq 1{,}4298 \text{ (0,25)}$$