UNIVERSITE ABBES LAGHROUR KHENCHELA

Faculté des sciences et techniques

Département : Génie Mécanique

Niveau : M1_CM — Examen final

Matière : Méthode des éléments finis — 24/05/2025

Corrigé type

<u>Exercice 01: (10 points)</u>

Tableau de connectivité

Eléments	Nœuds	
	i	j
1	1	2
2	2	3

La matrice de rigidité élémentaire

$$K_1 = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ 6l & 4l^2 & -6l & 2l^2 & 0 & 0 \\ -12 & -6l & 12 & -6l & 0 & 0 \\ 6l & 2l^2 & -6l & 4l^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_1 = \frac{EI}{l^3}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 6l & -12 & 6l \\ 0 & 0 & 6l & 4l^2 & -6l & 2l^2 \\ 0 & 0 & -12 & -6l & 12 & -6l \\ 0 & 0 & 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

La matrice de rigidité globale

$$K_g = K_{1+}K_2 = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ 6l & 4l^2 & -6l & 2l^2 & 0 & 0 \\ -12 & -6l & 24 & 0 & -12 & 6l \\ 6l & 2l^2 & 0 & 8l^2 & -6l & 2l^2 \\ 0 & 0 & -12 & -6l & 12 & -12l \\ 0 & 0 & 6l & 2l^2 & -12l & 4l^2 \end{bmatrix}$$

L’équation d’équilibre statique :

$$[K]\{u\} = \{F\}$$

$$\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ 6l & 4l^2 & -6l & 2l^2 & 0 & 0 \\ -12 & -6l & 24 & 0 & -12 & 6l \\ 6l & 2l^2 & 0 & 8l^2 & -6l & 2l^2 \\ 0 & 0 & -12 & -6l & 12 & -12l \\ 0 & 0 & 6l & 2l^2 & -12l & 4l^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \\ F_3 \\ M_3 \end{Bmatrix}$$

Les conditions aux limites :

$$v_1 = \theta_1 = v_3 = \theta_3 = 0$$

$$\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ 6l & 4l^2 & -6l & 2l^2 & 0 & 0 \\ -12 & -6l & 24 & 0 & -12 & 6l \\ 6l & 2l^2 & 0 & 8l^2 & -6l & 2l^2 \\ 0 & 0 & -12 & -6l & 12 & -12l \\ 0 & 0 & 6l & 2l^2 & -12l & 4l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ v_2 \\ \theta_2 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ -20 \\ 0 \\ F_3 \\ M_3 \end{Bmatrix}$$

Donc le système devient :

$$\frac{EI}{l^3}\begin{bmatrix} 24 & 0 \\ 0 & 8l^2 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -20 \\ 0 \end{Bmatrix}$$

$$\frac{EI}{l^3} 24 v_2 = -20$$

$$v_2 = -\frac{20l^3}{24EI} = -\frac{20 \times 2^3}{24 \times 210 \times 10^6 \times 6 \times 10^{-6}} = -0.0053m$$

$$\frac{EI}{l^3} 8l^2 \theta_2 = 0$$

$$\theta_2 = 0$$

Reviens au système initial :

$$\frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l & 0 & 0 \\ 6l & 4l^2 & -6l & 2l^2 & 0 & 0 \\ -12 & -6l & 24 & 0 & -12 & 6l \\ 6l & 2l^2 & 0 & 8l^2 & -6l & 2l^2 \\ 0 & 0 & -12 & -6l & 12 & -12l \\ 0 & 0 & 6l & 2l^2 & -12l & 4l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -0.0053 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ -20 \\ 0 \\ F_3 \\ M_3 \end{Bmatrix}$$

$$-12\frac{EI}{l^3}(0.0053) = F_1 \Rightarrow F_1 = -12\frac{210 \times 10^6 \times 6 \times 10^{-6}}{2^3}(-0.0053) = 0.017\ kN$$

$$-6l\frac{EI}{l^3}(0.0053) = M_1 \Rightarrow M_1 = -6 \times 2\frac{210 \times 10^6 \times 6 \times 10^{-6}}{2^3}(-0.0053) = 0.017\ kN.m$$

$$-12\frac{EI}{l^3}(0.0053) = F_3 \Rightarrow F_3 = -12\frac{210 \times 10^6 \times 6 \times 10^{-6}}{2^3}(-0.0053) = 0.017\ kN$$

$$6l\frac{EI}{l^3}(0.0053) = M_3 \Rightarrow M_3 = 6 \times 2\frac{210 \times 10^6 \times 6 \times 10^{-6}}{2^3}(-0.0053) = -0.017\ kN.m$$

Exercice 01: (10 points)

Tableau de connectivité

Eléments	Nœuds	
	i	j
❶	1	2
❷	2	3
❸	1	3

	C	*C*^{2}	*S*	*S*^{2}	*CS*	-*CS*
θ_1=0°	1	1	0	0	0	0
θ_2=135°	-0.71	0.5	0.71	0.5	-0.5	0.5
θ_3=90°	0	0	1	1	0	0

La matrice de rigidité élémentaire

$$K_1 = \frac{EA}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS & 0 & 0 \\ CS & S^2 & -CS & -S^2 & 0 & 0 \\ -C^2 & -CS & C^2 & CS & 0 & 0 \\ -CS & -S^2 & CS & S^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \frac{EA}{1}\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_2 = \frac{EA}{\sqrt{2}L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C^2 & CS & -C^2 & -CS \\ 0 & 0 & CS & S^2 & -CS & -S^2 \\ 0 & 0 & -C^2 & -CS & C^2 & CS \\ 0 & 0 & -CS & -S^2 & CS & S^2 \end{bmatrix} = \frac{EA}{\sqrt{2}L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5 & -0.5 & 0.5 \\ 0 & 0 & -0.5 & 0.5 & 0.5 & -0.5 \\ 0 & 0 & -0.5 & 0.5 & 0.5 & -0.5 \\ 0 & 0 & 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

$$K_3 = \frac{EA}{L}\begin{bmatrix} C^2 & CS & 0 & 0 & -C^2 & -CS \\ CS & S^2 & 0 & 0 & -CS & -S^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -C^2 & -CS & 0 & 0 & C^2 & CS \\ -CS & -S^2 & 0 & 0 & CS & S^2 \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

La matrice de rigidité globale

$$K_g = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1.35 & -0.35 & -0.35 & 0.35 \\ 0 & 0 & -0.35 & 0.35 & 0.35 & -0.35 \\ 0 & 0 & -0.35 & 0.35 & 0.35 & -0.35 \\ 0 & -1 & 0.35 & -0.35 & -0.35 & 1.35 \end{bmatrix}$$

L’équation d’équilibre statique :

$$[K]\{u\} = \{F\}$$

$$\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1.35 & -0.35 & -0.35 & 0.35 \\ 0 & 0 & -0.35 & 0.35 & 0.35 & -0.35 \\ 0 & 0 & -0.35 & 0.35 & 0.35 & -0.35 \\ 0 & -1 & 0.35 & -0.35 & -0.35 & 1.35 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \end{Bmatrix}$$

Les conditions aux limites :

$$u_1 = v_1 = u_3 = v_3 = 0$$

$$\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1.35 & -0.35 & -0.35 & 0.35 \\ 0 & 0 & -0.35 & 0.35 & 0.35 & -0.35 \\ 0 & 0 & -0.35 & 0.35 & 0.35 & -0.35 \\ 0 & -1 & 0.35 & -0.35 & -0.35 & 1.35 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ 0 \\ -10 \\ F_{x3} \\ F_{y3} \end{Bmatrix}$$

Donc le système devient :

$$\frac{210 \times 10^6 \times 0.001}{1}\begin{bmatrix} 1.35 & -0.35 \\ -0.35 & 0.35 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10 \end{Bmatrix}$$

$$210 \times 10^3 u_2 = -10 \Rightarrow u_2 = -0.048 \times 10^{-3} m$$

$$210 \times 10^3 [1.35(-0.048 \times 10^{-3}) - 0.35 v_2] = 0 \Rightarrow v_2 = -0.019 \times 10^{-3} m$$

Reviens au système initial :

$$\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1.35 & -0.35 & -0.35 & 0.35 \\ 0 & 0 & -0.35 & 0.35 & 0.35 & -0.35 \\ 0 & 0 & -0.35 & 0.35 & 0.35 & -0.35 \\ 0 & -1 & 0.35 & -0.35 & -0.35 & 1.35 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -0.048 \times 10^{-3} \\ -0.019 \times 10^{-3} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ 0 \\ -10 \\ F_{x3} \\ F_{y3} \end{Bmatrix}$$

$$F_{x1} = -1 \times \frac{210 \times 10^6 \times 0.001}{1}(-0.048 \times 10^{-3}) = 10.08\ kN$$

$$F_{y1} = 0\ kN$$

$$F_{x3} = \frac{210 \times 10^6 \times 0.001}{1}[-0.35(-0.048 \times 10^{-3}) + 0.35(-0.019 \times 10^{-3})] = 2.13\ kN$$

$$F_{y3} = \frac{210 \times 10^6 \times 0.001}{1}[0.35(-0.048 \times 10^{-3}) - 0.35(-0.019 \times 10^{-3})] = -2.13\ kN$$