University of Khenchela College of Sciences and Technology Department of Sciences and Technology	**Analysis 2**	**Year: 2024/2025** **Enseignant: F.Bekhouche**
	Final Exam 2	

Exercise 01

1. Give the Taylor expansion up to order 3 near 0 ($LD_3(0)$) of $f(x) = e^x \ln(1+x)$.
2. Deduce the value of $f''(0)$ and $f^{(3)}(0)$.
3. Use the first question to compute the following limit:

$$\lim_{x \to 0} \frac{e^x \ln(1+x) - x}{x^2}$$

Exercise 02

1. Decompose the following rational function into partial fractions of the form:

$$f(x) = \frac{1}{x(1+x^2)} = \frac{a}{x} + \frac{bx+c}{1+x^2},$$

where a, b, c are constants to be determined.

2. Find a primitive function of the following integral:

$$I = \int \frac{dx}{x(1+x^2)}$$

3. Deduce the following primitive:

$$I_0 = \int \frac{\arctan(x)}{x^2}\, dx$$

4. Use the result from question 2 to solve the following differential equation:

$$y' + \frac{y}{x} = -\frac{y^2}{1+x^2}, \quad y(1) = 2.$$

Exercise 03

We consider the differential equation:

$$y'' - 4y' + 3y = x^2 e^x \tag{E$_1$}$$

1. Solve the homogeneous differential equation associated with (E_1).
2. Find a particular solution (y_p) of (E_1), then give the general solution (y_g) of (E_1).
3. Determine the unique solution y of (E_1) satisfying the initial conditions

$$y(0) = 2 \text{ and } y(\ln(2)) = -\frac{1}{3}.$$

Good luck

University of Khenchela College of Sciences and Technology Department of Sciences and Technology	Analysis 2 Final Exam 2	Year: 2024/2025 Enseignant: F.Bekhouche

Exercise 1. (5.5 points)

1 The 3th order Taylor expansion near 0 of $f(x) = e^x \ln(1+x)$:
We have:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o(x^3) \quad (1 \text{ pt})$$

and

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) \quad (1.5 \text{ pt})$$

Hence:

$$\begin{aligned} f(x) &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)\right) \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} + x^2 - \frac{x^3}{2} + \frac{x^3}{2} + o(x^3) \\ &= x + \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) \quad (1 \text{ pt}). \end{aligned}$$

2 The goal is to identify the Taylor expansion of f using the Taylor-Young formula:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + o(x^3)$$

By uniqueness of the Taylor expansions and comparing the coefficients of x^2 and x^3, we obtain:

$$f''(0) = 1, \quad f^{(3)}(0) = 2 \quad (0.5 \text{ pt}) + (0.5 \text{ pt})$$

3 By replacing $f(x)$ with its Taylor expansion, we obtain:

$$\lim_{x\to 0} \frac{e^x \ln(1+x) - x}{x^2} = \lim_{x\to 0} \frac{x + \frac{x^2}{2} + o(x^2) - x}{x^2} = \frac{1}{2} \quad (1 \text{ pt})$$

Exercise 2. (9 points)

1. Partial Fraction Decomposition

We decompose:

$$\frac{1}{x(1+x^2)} = \frac{a}{x} + \frac{bx+c}{1+x^2}$$

Multiply both sides by $x(1+x^2)$:

$$1 = a(1+x^2) + x(bx+c)$$
$$= a + ax^2 + bx^2 + cx = (a+b)x^2 + cx + a$$

Equating coefficients:

$$\begin{cases} a + b = 0 \\ c = 0 \\ a = 1 \end{cases} \Rightarrow a = 1,\ b = -1,\ c = 0 \quad (1.5 \text{ pt})$$

Thus:

$$\frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}$$

2. Compute the Indefinite Integral

$$I = \int \frac{dx}{x(1+x^2)} = \int \left(\frac{1}{x} - \frac{x}{1+x^2} \right) dx$$

We integrate term by term:

$$\int \frac{1}{x} dx = \ln|x|, \quad \int \frac{x}{1+x^2} dx = \frac{1}{2}\ln(1+x^2)$$

Therefore:

$$I = \ln|x| - \frac{1}{2}\ln(1+x^2) + C \quad (1.5\text{ pt})$$

3. Evaluate the following primitive

$$I_0 = \int \frac{\arctan(x)}{x^2} dx$$

Integration by parts Let: (1 pt)

$$\begin{cases} v'(x) = \frac{1}{x^2} \Rightarrow v(x) = -\frac{1}{x} \\ u(x) = \arctan(x) \Rightarrow u'(x) = \frac{1}{1+x^2} \end{cases}$$

$$I_0 = -\frac{\arctan(x)}{x} + \int \frac{1}{x(1+x^2)} dx$$

From part 2:

$$\int \frac{1}{x(1+x^2)} dx = \ln|x| - \frac{1}{2}\ln(1+x^2)$$

So:

$$I_0 = -\frac{\arctan(x)}{x} + \ln|x| - \frac{1}{2}\ln(1+x^2). \quad (1\text{ pt})$$

4. Solve the Differential Equation

Given:

$$y' + \frac{y}{x} = -\frac{y^2}{1+x^2}, \quad y(1) = 2$$

Let $z = \frac{1}{y} \Rightarrow y' = -\frac{z'}{z^2}$ Substitute:

$$-z' + \frac{z}{x} = -\frac{1}{1+x^2} \Rightarrow z' - \frac{z}{x} = \frac{1}{1+x^2} \quad (1\text{ pt})$$

This is a linear ODE. Integrating factor:

$$z_h = k\exp\left(\int \frac{1}{x} dx\right) = kx \quad (1\text{ pt})$$

where $k \in \mathbb{R}$
Now, assume that $z_p = k(x)x$,
(0.5 pt) So, we get

$$k'(x)x = \frac{1}{1+x^2} \Rightarrow k'(x) = \frac{1}{x(1+x^2)}$$

Integrate:

$$k(x) = \ln|x| - \frac{1}{2}\ln(1+x^2) \Rightarrow z_p = x\ln|x| - \frac{x}{2}\ln(1+x^2)) \quad (0.5\text{ pt})$$

$$\Rightarrow z_g = kx + x\ln|x| - \frac{x}{2}\ln(1+x^2) \quad (0.5\text{ pt})$$

Since $y = \frac{1}{z}$, so

$$y_g = \frac{1}{kx + x\ln|x| - \frac{x}{2}\ln(1+x^2)}$$

Apply initial condition $y(1) = 2$ to find k:

$$y(1) = 2 \Rightarrow \frac{1}{k - \frac{1}{2}\ln(2)} = 2 \Rightarrow K = \frac{1}{2}(1 + \ln 2)$$

Thus the particular solution is:

$$y(x) = \frac{1}{\frac{1}{2}(1+\ln 2)x + x\ln|x| - \frac{x}{2}\ln(1+x^2)} \quad \text{(0.5 pt)}$$

Exercise. (5.5 points) Consider the differential equation:

$$y'' - 4y' + 3y = x^2 e^x \tag{E$_1$}$$

1 **Solve the homogeneous differential equation associated with (E$_1$).**

We solve the homogeneous equation:

$$y'' - 4y' + 3y = 0$$

The characteristic equation is:

$$r^2 - 4r + 3 = 0 \Rightarrow (r-3)(r-1) = 0 \quad \text{(0.5 pt)}$$

$$r_1 = 3, \quad r_2 = 1 \quad \text{(0.5 pt)}$$

Hence, the general solution to the homogeneous equation is:

$$y_h(x) = C_1 e^{3x} + C_2 e^x \quad \text{(1 pt)}$$

2 **Find a particular solution** (y_p) **of (E$_1$), then give the general solution** (y_g) **of (E$_1$).**

The nonhomogeneous term is x^2e^x, and since e^x is a solution to the homogeneous equation (root $r = 1$), we multiply by x. We propose:

$$y_p(x) = x(Ax^2 + Bx + C)e^x = (Ax^3 + Bx^2 + Cx)e^x \quad \text{(0.5 pt)}$$

Now substitute compute derivatives into the differential equation:

$$y_p'' - 4y_p' + 3y_p = x^2e^x$$

Using:

$$y_p = (Ax^3+Bx^2+Cx)e^x y_p' = (Ax^3+(3A+B)x^2+(2B+C)x+C)e^x y_p'' = (Ax^3+(6A+B)x^2+(6A+4B+C)x+(2B+2C))e^x$$

So set:

$$\begin{cases} A - 4A + 3A = 0 \\ (6A+B) - 12A - 4B + 3B = 0 \Rightarrow -6A = 1 \Rightarrow A = -\frac{1}{6} \\ (6A+4B+C) - 8B - 4C + 3C = 0 \Rightarrow 6A - 4B = 0 \Rightarrow B = \frac{3A}{2} = -\frac{1}{4} \\ (2B+2C) - 4C = 0 \Rightarrow 2B - 2C = 0 \Rightarrow C = B = -\frac{1}{4} \end{cases}$$

Thus:

$$A = -\frac{1}{6}, \quad B = -\frac{1}{4}, \quad C = -\frac{1}{4} \quad \text{(1 pt)}$$

Hence the particular solution is:

$$y_p(x) = x\left(-\frac{1}{6}x^2 - \frac{1}{4}x - \frac{1}{4}\right)e^x = \left(-\frac{1}{6}x^3 - \frac{1}{4}x^2 - \frac{1}{4}x\right)e^x \quad \text{(0.5 pt)}$$

The general solution is:

$$y_g(x) = C_1e^{3x} + C_2e^x + \left(-\frac{1}{6}x^3 - \frac{1}{4}x^2 - \frac{1}{4}x\right)e^x \quad \text{(0.5 pt)}$$

3 **Determine the unique solution** y **of (E$_1$) satisfying the initial conditions** $y(0) = 2$ **and** $y(\ln(2)) = -\frac{1}{3}$**.**

We compute:

$$y(0) = C_1 + C_2 + 0 = 2 \Rightarrow C_1 + C_2 = 2 \tag{1}$$

Now compute $y(1)$:

$$y(\ln(2)) = 8C_1 + 2C_2 + 2\left(-\frac{1}{6} - \frac{1}{4} - \frac{1}{4}\right) = 8C_1 + 2C_2 - \left(\frac{4}{3}\right)e = -\frac{1}{3} \tag{2}$$

So:

$$8C_1 + 2C_2 = 1$$

Now solve the system:

$$\begin{cases} C_1 + C_2 = 2 \\ 8C_1 + 2C_2 = 1 \end{cases}$$

Thus $C_1 = -\dfrac{1}{2},\ c_2 = \dfrac{5}{2}$ Hence the the unique solution is:

$$y(x) = -\frac{1}{2}e^{3x} + \frac{5}{2}e^{x} + \left(-\frac{1}{6}x^3 - \frac{1}{4}x^2 - \frac{1}{4}x\right)e^{x} \quad \text{(1 pt)}$$