Abbes Laghrour University of Khenchela
Faculty of Science and Technology
Module: Mathematics 4

Level: 2^{nd} $GC-GM$
Duration: $1h30min$
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Exam solution

Exercise 1 (04 pts)**.**

1. $z^6+1=i\sqrt{3}$.
   $z^6=-1+i\sqrt{3}$,
   Moivre's formula $z=(-1+i\sqrt{3})^{\frac{1}{6}}=(r(\cos\theta+i\sin\theta))^{\frac{1}{6}}=r^{\frac{1}{6}}\left(\cos\dfrac{\theta+2\pi k}{6}+i\sin\dfrac{\theta+2\pi k}{6}\right)$

   $r=\sqrt{x^2+y^2}=\sqrt{(-1)^2+(\sqrt{3})^2}=2.$

$$\begin{cases}\cos &= \frac{x}{r}=\frac{-1}{2}\\ \sin &= \frac{y}{r}=\frac{\sqrt{3}}{2}.\end{cases}\Rightarrow\theta=\frac{2\pi}{3}+2\pi k$$

   $z=2^{\frac{1}{6}}\left(\cos(\frac{\pi}{9}+\frac{\pi k}{3})+i\sin(\frac{\pi}{9}+\frac{\pi k}{3})\right),\quad k=0,1,2.$
   if $k=0\Rightarrow z_0=2^{\frac{1}{6}}\left(\cos(\frac{\pi}{9}+\frac{\pi k}{3})+i\sin(\frac{\pi}{9}+\frac{\pi k}{3})\right).$
   if $k=1\Rightarrow z_1=2^{\frac{1}{6}}\left(\cos(\frac{\pi}{9})+i\sin(\frac{\pi}{9})\right).$
   if $k=2\Rightarrow z_2=2^{\frac{1}{6}}\left(\cos(\frac{4\pi}{9})+i\sin(\frac{4\pi}{9})\right).$

2. $z^3-3z^2+3z+1=0.$

   $z^3-3z^2+3z+1=(z-1)^3+2$ *so by writing* $(z-1)^3+2=0$ *by writing in polar form*

$$r=|2|$$

$$\begin{cases}\cos &= \frac{x}{r}=\frac{-2}{2}=-1\\ \sin &= \frac{y}{r}=0\end{cases}\Rightarrow\theta=\pi+2\pi k.$$

   $(z-1)^3=2\left(\cos(\pi+2\pi k)+i\sin(\pi+2\pi k)\right).$
   $z-1=2^{\frac{1}{3}}\left(\cos(\pi+2\pi k)+i\sin(\pi+2\pi k)\right)^{\frac{1}{3}}$
   $z=2^{\frac{1}{3}}\left(\cos(\pi+2\pi k)+i\sin(\pi+2\pi k)\right)^{\frac{1}{3}}+1,\qquad k=0,1,2.$
   if $z_0=2^{\frac{1}{3}}(\frac{1}{2}+i\frac{\sqrt{3}}{2})+1.$
   if $z_1=-2^{\frac{1}{3}}+1.$
   if $z_2=2^{\frac{1}{3}}\frac{1}{2}-i\frac{\sqrt{3}}{2})+1.$

Exercise 2 (06 pts)**.**

1. *The function u defined below is a harmonic function: $\Delta u = 0$.*

$$u(x, y) = y\ \cos y\, chx + x\ \sin y\, shx, \quad x, y \in \mathbb{R},$$
$$\Delta u = 0 \Leftrightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$
$$\frac{\partial^2 u}{\partial x^2} = (y\cos y + 2\sin y)\cosh x + s\sin y\sinh x,$$
$$\frac{\partial^2 u}{\partial y^2} = (-\sin y - \sin y - y\cos y)\cosh x - x\sin y\sinh x.$$

2. *$v =$? The Cauchy-Riemann equations are written:*

$$\begin{cases} \frac{\partial v}{\partial y} &= \frac{\partial u}{\partial x} = (y\cos y + \sin y)\sinh x + x\sin y\cosh x. \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} = (-\cos y + y\sin y)\cosh x - x\cos y\sinh x. \end{cases}$$

By integrating (by parts) the equation with respect to y

$$v = y\sin y\sinh x - x\cos y\cosh x + C(x),$$

By deriving the equation and comparing

$$v = y\sin y\sinh x - x\cos y\cosh x + C, \qquad C \in \mathbb{R}.$$

Exercise 3 (07 pts)**.**

$$\int_{(0,3)}^{(2,4)} (2y + x^2)dx + (3x - y)dy. \tag{1}$$

the length of:

a) The parabola $x = 2t$, $y = t^2 + 3$,

$$\int_{t=0}^{t=1} (24t^2 + 12 - 2t^3 - 6t)dt = \frac{33}{2}. \tag{2}$$

b) The broken line formed by the line segments $(0, 3)$ à $(2, 3)$ and $(2, 3)$ à $(2, 4)$,

$$\int_{x=0}^{2} (6 + x^2 dx) + (3x - 3)0 = \frac{44}{3}, \tag{3}$$

$$\int_{y=3}^{4} (2y + 4)0 + (6 - y)dy = \frac{5}{2}. \tag{4}$$

result is therefore $\frac{44}{3} + \frac{5}{2} = \frac{103}{6}$.

c) The line segment with endpoints $(0, 3)$ and $(2, 4)$
An equation of the line joining $(0; 3)$ ***to*** $(2; 4)$ ***is*** $x = 2y - 6$
$\int_{y=3}^{4} (8y^2 - 39y + 54)dy = \frac{97}{6}$.

Exercise 4 (03 pts)**.**

1.

$$f(z) = \frac{1}{z^2} e^{\frac{1}{z}}.$$

$$u = \frac{1}{z} \Rightarrow z = \frac{1}{u}$$

$$\frac{1}{z^2} e^{\frac{1}{z}} = u^2 e^u = u^2 \left(1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots \right)$$

$$= u^2 + u^3 + \frac{u^4}{2!} + \frac{u^5}{3} + \cdots$$

$$= \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{2! z^4} + \frac{1}{3! z^5} + \cdots .$$

2. The principal part of the Laurent series has an infinity of terms, so the point $z = 0$ is an essential singularity.

3. $Res(f; 0) = 0$.
$Res(f; 0) = 0$ *because the Laurent series for $f(z)$ does not contain the term $\frac{1}{z}$.*

Good luck