Geometry

Final Exam

Durée : 1h.30

Exercice n=1 — 7 points

Let X be a set. Suppose there exists a mapping :

$$\Phi : \begin{cases} X \times X \longrightarrow \vec{X} \\ (A, B) \longmapsto \Phi(A, B) \end{cases}$$

such that :

(i') $\forall A, B, C \in X$, we have $\Phi(A, B) + \Phi(B, C) = \Phi(A, C)$.

(ii') $\forall M \in X$, the mapping $\Phi_M : P \longmapsto \Phi_M(P) = \Phi(M, P)$ is a bijection from X to $\vec{X}$.

For $M \in X$ and $\overrightarrow{v} \in \overrightarrow{X}$, define $M \widehat{+} \overrightarrow{v}$ by : $M \widehat{+} \overrightarrow{v} = (\Phi_M)^{-1}(\overrightarrow{v})$.

1° Show that this definition is well-defined and that $\widehat{+}$ is an external composition law on X.

2° Show that $(X, \overrightarrow{X}, \widehat{+})$ is an affine space.

Exercice n=2 — 6 points

Let $f \in \mathscr{A}(E, E')$ with φ its linear part. Let F be an affine subspace of E with direction $\overrightarrow{F}$, and let G' be an affine subspace of E' with direction $\overrightarrow{G'}$. Show that :

(i) $f(F)$ is an affine subspace of E' with direction $\varphi(\overrightarrow{F})$,

(ii) $f^{-1}(G')$ is either empty or an affine subspace of E with direction $\varphi^{-1}(\overrightarrow{G'})$.

Exercice n=3 — 7 points

Let E be a finite-dimensional Euclidean space and F_1, F_2 be two subspaces of E. Prove that :

$$(F_1 \cap F_2)^{\perp} = F_1^{\perp} + F_2^{\perp},$$
$$(F_1 + F_2)^{\perp} = F_1^{\perp} \cap F_2^{\perp}.$$

Correction of Final Exam

Exercice n=1 **7 points**

1° (2 points) We must verify that the definition of $M\hat{+}\vec{v}$ makes sense for all $M \in X$ and $\vec{M} \in \vec{X}$. Hypothesis (ii') states that the mapping Φ_A is bijective for all M, so Φ_M^{-1} is well-defined as a mapping from $\vec{X}$ to X (since Φ maps X to $\vec{X}$). Thus, $\Phi_M^{-1}(\vec{u})$ is indeed a well-defined element of X for all M and $\vec{v}$. We can therefore assert that the proposed definition is meaningful.
Next, we show that $\hat{+}$ is an external composition law. This has already been verified, since we have shown that for all $M \in X$ and $\vec{v} \in \vec{X}$, $M\hat{+}\vec{v}$ is a uniquely determined element of X. Thus, $\hat{+}$ is indeed an external composition law that assigns to each pair $(M,\vec{v})$ in $X \times \vec{X}$ an element of X.
2° (5 points) We must show that $\hat{+}$ satisfies the three properties (i), (ii), and (iii) in the Definition of affine space.
- For (i), we must show that $A\hat{+}\vec{0} = A$ for all $A \in X$. By definition, $A\hat{+}\vec{0} = \Phi_A^{-1}(\vec{0})$, which, by the definition of an inverse mapping and hypothesis (ii'), is an element B of X such that $\vec{0} = \Phi_A(B) = \Phi(A,B)$. Applying (i') with $A = B = C$, we have $\Phi(A,A) + \Phi(A,A) = \Phi(A,A)$, so $\Phi(A,A) = \vec{0}$. But $\Phi(A,A) = \Phi_A(A)$. Thus, we have found two points, A and B, such that $\vec{0} = \Phi_A(A) = \Phi_A(B)$. Since Φ_A is bijective by (ii'), we conclude that $A = B$ and $A\hat{+}\vec{0} = A$.
- For (ii) : Let $A \in X$ and $\vec{u},\vec{v}$ be two vectors in $\vec{X}$. Define $B = A\hat{+}\vec{u}$, $C = B\hat{+}\vec{v}$, and $D = A\hat{+}(\vec{u}+\vec{v})$. We must show that $(A\hat{+}\vec{u})\hat{+}\vec{v} = A\hat{+}(\vec{u}+\vec{v})$, i.e., $C = D$.
By definition, $D = \Phi_A^{-1}(\vec{u}+\vec{v})$, so $\Phi_A(D) = \Phi(A,D) = \vec{u}+\vec{v}$.
On the other hand, $B = \Phi_A^{-1}(\vec{u})$, so $\vec{u} = \Phi_A(B) = \Phi(A,B)$, and $C = \Phi_B^{-1}(\vec{v})$, so $\vec{v} = \Phi_B(C) = \Phi(B,C)$. Thus, $\vec{u}+\vec{v} = \Phi(A,B)+\Phi(B,C) = \Phi(A,C)$ (by applying hypothesis (i')). We therefore have $\vec{u}+\vec{v} = \Phi(A,C) = \Phi(A,D)$, so $\Phi_A(C) = \Phi_A(D)$. Since Φ_A is bijective by (ii'), it follows that $C = D$.
For (iii), there is almost nothing to prove, since for all $A \in X$ and $\vec{u} \in \vec{E}$, $A\hat{+}\vec{u} = \Phi_A^{-1}(\vec{u})$. Thus, the mapping $\vec{u} \longmapsto A\hat{+}\vec{u}$ is simply the inverse mapping Φ_A^{-1}, which is bijective as the inverse of a bijection.
Therefore, $(X,\vec{X},\hat{+})$ is an affine space over $\vec{X}$.

Exercice n=2 **6 points**

(i) (3 points) Let $A \in F$. Define $F' = f(A) + \varphi(\overrightarrow{F})$. Since $\varphi(\overrightarrow{F})$ is a vector subspace of $\overrightarrow{E'}$ (by properties of linear maps), F' is an affine subspace of E'. We will show that $f(F) = F'$, which proves that $f(F)$ is an affine subspace.
Let $M' \in f(F)$. There exists $M = A + \overrightarrow{u} \in A + \overrightarrow{F} = F$ such that $M' = f(M) = f(A) + \varphi(M-A)$. Since $M - A \in \overrightarrow{F}$, we have $\varphi(M-A) \in \varphi(\overrightarrow{F})$, and thus $M' \in F' = f(A)+\varphi(\overrightarrow{F})$. This proves the inclusion

$f(F) \subset F'$.
Conversely, if $M' \in F' = f(A) + \varphi(\overrightarrow{F})$, we can write $M' = f(A) + \overrightarrow{v}$ with $\overrightarrow{u} \in \varphi(\overrightarrow{F})$. Therefore, there exists $\overrightarrow{u} \in \overrightarrow{F}$ such that $\overrightarrow{v} = \varphi(\overrightarrow{u})$. Let $M = A + \overrightarrow{u}$. Then $M \in A + \overrightarrow{F} = F$, and we have $f(M) = f(A) + \varphi(\overrightarrow{AM}) = f(A) + \varphi(\overrightarrow{u}) = M'$. Thus, $M' \in f(F)$, proving the inclusion $F' \subset f(F)$.
(ii) (3 points) Suppose $G = f^{-1}(G')$ is non-empty, and let $A \in f^{-1}(G')$. Then $f(A) \in G'$, and $G' = f(A) + \overrightarrow{G'}$. Note that $\overrightarrow{G} = \varphi^{-1}(\overrightarrow{G'})$ is always a vector subspace of $\overrightarrow{E}$. We will show that $G = A + \overrightarrow{G}$.
Let $M \in G$. Since $f(M) \in G'$, we have $f(M) = f(A) + \overrightarrow{v}$ with $\overrightarrow{v} \in \overrightarrow{G}$. This implies $\varphi(\overrightarrow{AM}) = \overrightarrow{v} \in \overrightarrow{G}$, so $\overrightarrow{AM} \in \varphi^{-1}(\overrightarrow{G'}) = \overrightarrow{G}$. Thus, $M = A + \overrightarrow{AM} \in A + \overrightarrow{G}$, proving the inclusion $G \subset A + \overrightarrow{G}$.
Now, let $M \in A + \overrightarrow{G}$. Then $\overrightarrow{AM} \in \overrightarrow{G} = \varphi^{-1}(\overrightarrow{G'})$, so $\varphi(\overrightarrow{AM}) \in \overrightarrow{G'}$. Hence, $f(M) = f(A) + \varphi(\overrightarrow{AM}) \in f(A) + \overrightarrow{G'} = G'$, meaning $M \in G$. This proves the inclusion $A + \overrightarrow{G} \subset G$.
We conclude that $G = A + \overrightarrow{G}$, so $G = f^{-1}(G')$ is indeed an affine subspace.

Exercice n=3 **7 points**

1. (5 points) Through the fact that

$$F_1 \cap F_2 \subset F_1, F_2 \implies \overset{\perp}{F_1}, \overset{\perp}{F_2} \subset (F_1 \cap F_2)^{\perp}$$

and $\overset{\perp}{F_1} + \overset{\perp}{F_2}$ is the smallest subspace containing both $\overset{\perp}{F_1}$ and $\overset{\perp}{F_2}$. This yields

$$\overset{\perp}{F_1} + \overset{\perp}{F_2} \subset (F_1 \cap F_2)^{\perp}.$$

Since

$$\overset{\perp}{F_1}, \overset{\perp}{F_2} \subset \overset{\perp}{F_1} + \overset{\perp}{F_2}$$

implies

$$\left(\overset{\perp}{F_1} + \overset{\perp}{F_2}\right)^{\perp} \subset F_1 \cap F_2.$$

Therefore

$$(F_1 \cap F_2)^{\perp} \subset \overset{\perp}{F_1} + \overset{\perp}{F_2},$$

hence

$$(F_1 \cap F_2)^{\perp} = \overset{\perp}{F_1} + \overset{\perp}{F_2}.$$

2. (2 points) The second identity is simply proved by substituting F_1 and F_2 in the first identity by $\overset{\perp}{F_1}$ and $\overset{\perp}{F_2}$.