

Exercice n=1

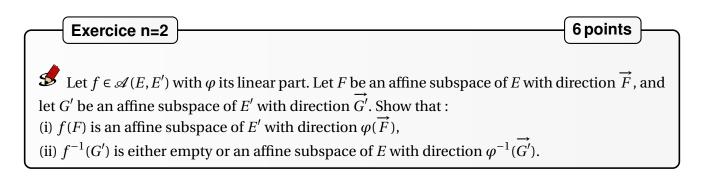
7 points

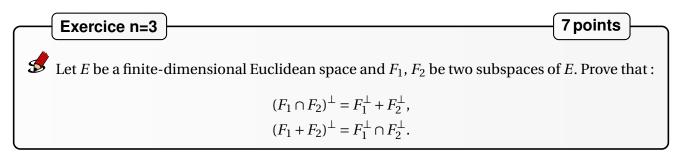
⁵⁵ Let *X* be a set. Suppose there exists a mapping :

$$\Phi: \begin{cases} X \times X \longrightarrow \vec{X} \\ (A, B) \longmapsto \Phi(A, B) \end{cases}$$

such that :

- (i') $\forall A, B, C \in X$, we have $\Phi(A, B) + \Phi(B, C) = \Phi(A, C)$.
- (ii') $\forall M \in X$, the mapping $\Phi_M : P \mapsto \Phi_M(P) = \Phi(M, P)$ is a bijection from X to \vec{X} .
- For $M \in X$ and $\vec{v} \in \vec{X}$, define $M \neq \vec{v}$ by $: M \neq \vec{v} = (\Phi_M)^{-1}(\vec{v})$.
- 1° Show that this definition is well-defined and that $\hat{+}$ is an external composition law on *X*.
- 2° Show that $(X, \vec{X}, \hat{+})$ is an affine space.





7 points



Exercice n=1

 \mathfrak{S} 1° (2 points) We must verify that the definition of $M + \vec{v}$ makes sense for all $M \in X$ and $\vec{M} \in \vec{X}$. Hypothesis (*ii'*) states that the mapping Φ_A is bijective for all M, so Φ_M^{-1} is well-defined as a mapping from \vec{X} to X (since Φ maps X to \vec{X}). Thus, $\Phi_M^{-1}(\vec{u})$ is indeed a well-defined element of X for all M and \vec{v} . We can therefore assert that the proposed definition is meaningful. Next, we show that $\hat{+}$ is an external composition law. This has already been verified, since we have shown that for all $M \in X$ and $\vec{v} \in \vec{X}$, $M \neq \vec{v}$ is a uniquely determined element of X. Thus, \neq is indeed an external composition law that assigns to each pair (M, \vec{v}) in $X \times \vec{X}$ an element of X. 2° (5 points) We must show that $\hat{+}$ satisfies the three properties (i), (ii), and (iii) in the Definition of affine space. - For (*i*), we must show that $A \neq \vec{0} = A$ for all $A \in X$. By definition, $A \neq \vec{0} = \Phi_A^{-1}(\vec{0})$, which, by the definition of an inverse mapping and hypothesis (*ii'*), is an element B of X such that $\vec{0} = \Phi_A(B) =$ $\Phi(A, B)$. Applying (i') with A = B = C, we have $\Phi(A, A) + \Phi(A, A) = \Phi(A, A)$, so $\Phi(A, A) = \vec{0}$. But $\Phi(A, A) = \Phi_A(A)$. Thus, we have found two points, A and B, such that $\vec{0} = \Phi_A(A) = \Phi_A(B)$. Since Φ_A is bijective by (ii'), we conclude that A = B and $A \neq \vec{0} = A$. - For (*ii*) : Let $A \in X$ and \vec{u}, \vec{v} be two vectors in \vec{X} . Define $B = A + \vec{u}, C = B + \vec{v}$, and $D = A + (\vec{u} + \vec{v})$. We must show that $(A + \vec{u}) + \vec{v} = A + (\vec{u} + \vec{v})$, i.e., C = D. By definition, $D = \Phi_A^{-1}(\vec{u} + \vec{v})$, so $\Phi_A(D) = \Phi(A, D) = \vec{u} + \vec{v}$. On the other hand, $B = \Phi_A^{-1}(\vec{u})$, so $\vec{u} = \Phi_A(B) = \Phi(A, B)$, and $C = \Phi_B^{-1}(\vec{v})$, so $\vec{v} = \Phi_B(C) = \Phi(B, C)$. Thus, $\vec{u} + \vec{v} = \Phi(A, B) + \Phi(B, C) = \Phi(A, C)$ (by applying hypothesis (*i'*)). We therefore have $\vec{u} + \vec{v} =$ $\Phi(A, C) = \Phi(A, D)$, so $\Phi_A(C) = \Phi_A(D)$. Since Φ_A is bijective by (ii'), it follows that C = D. For (*iii*), there is almost nothing to prove, since for all $A \in X$ and $\vec{u} \in \vec{E}$, $A + \vec{u} = \Phi_A^{-1}(\vec{u})$. Thus, the mapping $\vec{u} \mapsto A + \vec{u}$ is simply the inverse mapping Φ_A^{-1} , which is bijective as the inverse of a bijection. Therefore, $(X, \vec{X}, \hat{+})$ is an affine space over \vec{X} .

Exercice n=2

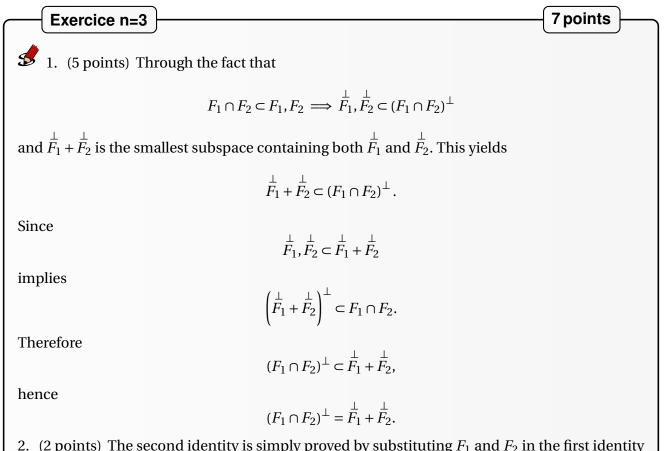
6 points

(i) (3 points) Let $A \in F$. Define $F' = f(A) + \varphi(\vec{F})$. Since $\varphi(\vec{F})$ is a vector subspace of $\vec{E'}$ (by properties of linear maps), F' is an affine subspace of E'. We will show that f(F) = F', which proves that f(F) is an affine subspace.

Let $M' \in f(F)$. There exists $M = A + \vec{u} \in A + \vec{F} = F$ such that $M' = f(M) = f(A) + \varphi(M - A)$. Since $M - A \in \vec{F}$, we have $\varphi(M - A) \in \varphi(\vec{F})$, and thus $M' \in F' = f(A) + \varphi(\vec{F})$. This proves the inclusion

$f(F) \subset F'$.

Conversely, if $M' \in F' = f(A) + \varphi(\vec{F})$, we can write $M' = f(A) + \vec{v}$ with $\vec{u} \in \varphi(\vec{F})$. Therefore, there exists $\vec{u} \in \vec{F}$ such that $\vec{v} = \varphi(\vec{u})$. Let $M = A + \vec{u}$. Then $M \in A + \vec{F} = F$, and we have $f(M) = f(A) + \varphi(\vec{AM}) = f(A) + \varphi(\vec{u}) = M'$. Thus, $M' \in f(F)$, proving the inclusion $F' \subset f(F)$. (ii) (3 points) Suppose $G = f^{-1}(G')$ is non-empty, and let $A \in f^{-1}(G')$. Then $f(A) \in G'$, and $G' = f(A) + \vec{G'}$. Note that $\vec{G} = \varphi^{-1}(\vec{G'})$ is always a vector subspace of \vec{E} . We will show that $G = A + \vec{G}$. Let $M \in G$. Since $f(M) \in G'$, we have $f(M) = f(A) + \vec{v}$ with $\vec{v} \in \vec{G}$. This implies $\varphi(\vec{AM}) = \vec{v} \in \vec{G}$, so $\vec{AM} \in \varphi^{-1}(\vec{G'}) = \vec{G}$. Thus, $M = A + \vec{AM} \in A + \vec{G}$, proving the inclusion $G \subset A + \vec{G}$. Now, let $M \in A + \vec{G}$. Then $\vec{AM} \in \vec{G} = \varphi^{-1}(\vec{G'})$, so $\varphi(\vec{AM}) \in \vec{G'}$. Hence, $f(M) = f(A) + \varphi(\vec{AM}) \in f(A) + \vec{G'} = G'$, meaning $M \in G$. This proves the inclusion $A + \vec{G} \subset G$. We conclude that $G = A + \vec{G}$, so $G = f^{-1}(G')$ is indeed an affine subspace.



2. (2 points) The second identity is simply proved by substituting F_1 and F_2 in the first identity by $\stackrel{\perp}{F_1}$ and $\stackrel{\perp}{F_2}$.