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Level: 1st informatics
Subject: Analysis I

A correction of the first exam

Exercise01 (05pts)

Consider the function $f: [-1;1] \to [0;1]$ defined by $f(x) = \sin(\arccos(x))$

1- Calculating $f(0), f(1)$ *and* $f(-1)$.

$f(0) = \sin(\arccos(0)) = \sin\left(\frac{\pi}{2}\right) = 1$ 01

$f(1) = \sin(\arccos(1)) = \sin(0) = 0$

$f(-1) = \sin(\arccos(-1)) = \sin(-\pi) = 0$

2- Finding $f'(x)$ for $x \in]-1;1[$ then sketch the table of variation of the function f.

Since sine and arcos are differentiable over $\mathbb{R}$, $]-1;1[$ respectively, we thus conclude that the composite of both functions f is differentiable on $]-1;1[$. 01

$$f'(x) = [\arccos(x)]' \times \sin'(\arccos(x)) = -\frac{1}{\sqrt{1-x^2}} \times \cos(\arccos(x))$$

$$f'(x) = -\frac{x}{\sqrt{1-x^2}}$$

$f'(x) \geq 0$ *when* $x \in]-1;0]$. *Then* f *is increasing on* $]-1;0]$.

$f'(x) < 0$ *when* $x \in]0;1[$. *Then* f *is strictly deccreasing on* $]0;1[$. 01

x	-1		0		1
$f'(x)$		+	0	-	
$f(x)$	0	↗	1	↘	0

3- Showing that $\forall x \in [-1;1]$ $\sin(\arccos(x)) = \sqrt{1-x^2}$

We know that $\sin^2(\arccos(x)) + \cos^2(\arccos(x)) = 1$.

Then $\sin^2(\arccos(x)) + x^2 = 1$. 01

$\forall x \in [-1;1]$ $\sin(\arccos(x)) = \sqrt{1-x^2}$.

4- Deducing that the graph of f is the top half of unit circle.

We have found that $f(x) = \sqrt{1-x^2}$. Then $y = \sqrt{1-x^2}, y \geq 0$ 01

So $x^2 + y^2 = 1$ with $y \geq 0$. This is the top half of unit circle.

Exercise02 (08pts)

1. Determine the reel constants a *and* b such that $\frac{9}{x^2-5x-14} = \frac{a}{x+2} + \frac{b}{x-7}$.

Using identification gives $a = -1$ *and* $b = 1$. 02

Therefore $\frac{9}{x^2-5x-14} = -\frac{1}{x+2} + \frac{1}{x-7}$

2. Find the indefinite integral $\int \frac{9}{x^2-5x-14}dx$, then $\int_0^1 \frac{9}{x^2-5x-14}dx$.

$$\int \frac{9}{x^2-5x-14}\,dx = \int\left(-\frac{1}{x+2}+\frac{1}{x-7}\right)dx = ln|x-7| - ln|x+2| + c$$ 01

Thus $\int \frac{9}{x^2-5x-14}\,dx = ln\left|\frac{x-7}{x+2}\right| + c.$

We thus deduce that $\int_0^1 \frac{9}{x^2-5x-14}dx = \left[ln\left|\frac{x-7}{x+2}\right|\right]_0^1 = \ln 2 - \ln\frac{7}{2} = \ln\frac{4}{7}.$ 01

3. Using a suitable change of variable to evaluate $\int_0^{\frac{\pi}{2}} \frac{9\cos t}{-14-5\sin t+\sin^2 t}dt.$

Taking $x = \sin t$ yields $\begin{cases} dx = \cos t\, dt \\ x = 0 \text{ if } t = 0 \\ x = 1 \text{ if } t = \frac{\pi}{2} \end{cases}.$ 01

$$\int_0^{\frac{\pi}{2}} \frac{9\cos t}{-14-5\sin t\ +\sin^2 t}dt = \int_0^1 \frac{9}{x^2-5x-14}dx = \ln\frac{4}{7}.$$ 01

4. Let $x \in]7;+\infty[$, solve the following first order differential equation:

$$y' - \frac{9}{x^2-5x-14}y = \frac{x-7}{x^2-5x-14}$$

- Integrating factor $r(x) = e^{\int -\frac{9}{x^2-5x-14}dx} = e^{-ln\left|\frac{x-7}{x+2}\right|} = \frac{x+2}{x-7}.$
- Multiplying by $\frac{x-7}{x+2}$ both sides of DE gives $\frac{x+2}{x-7}y' - \frac{9}{(x-7)^2}y = \frac{1}{x-7}.$

  That is $\left(\frac{x+2}{x-7}y\right)' = \frac{1}{x-7}.$
- integrating both sides we get $\frac{x+2}{x-7}y = \ln(x-7) + c.$

  Hence $y = (\ln(x-7)+c)\frac{x-7}{x+2}\ \ /c \in \mathbb{R}.$

02

Exercise03 ***(07pts)***

Consider the following second order differential equation:

$$y'' - 4y' + 4y = (2x-4)e^x \ \ldots.(2)$$

1. Solving the homogeneous differential equation given by: $y'' - 4y' + 4y = 0.$
   The auxiliary equation is $r^2 - 4r + 4 = 0$ has a repeated root $r = 2$.
   Therefore the complementary solution is going to be:
   $$y_{c=}(kx+k')e^{2x} \text{ where } k, k' \in \mathbb{R}.$$
   02,5
2. Determining the constants α and β so that $y_{p=}(\alpha x+\beta)e^x$ is a particular solution
   $$y_{p=}(\alpha x+\beta)e^x \Rightarrow y'_{p=}(\alpha x+\alpha+\beta)e^x \Rightarrow y''_{p=}(\alpha x+2\alpha+\beta)e^x.$$ 01,5
   As y_p is a solution of (2), we get
   $(\alpha x+2\alpha+\beta)e^x - 4(\alpha x+\alpha+\beta)e^x + 4(\alpha x+\beta)e^x = (2x-4)e^x$ 01,5
   Hence $(\alpha x - 2\alpha+\beta)e^x = (2x-4)e^x.$
   Using identification, we find $\alpha = 2$ and $\beta = 0$.
   So $y_{p=}2xe^x.$
3. Find the general solution of (2).
   Knowing that $y = y_p + y_c$, we thus write
   $$y = 2xe^x + (kx+k')e^{2x}\ /\ k, k' \in \mathbb{R}$$ 01,5