Abbes Laghrour University of Khenchela | May 2025
Faculty of Science and Technology | Duration ; 1h30min
Department of Science and Technology Subject: Mathematics 2

Correction of the final exam

Exercise 1.(4.5 pts)

Calculate the following integrals

$$I_1 = \int_1^4 \frac{\sqrt{x}-1}{\sqrt{x}} dx, \qquad I_2 = \int_0^{\frac{\pi}{2}} \sin(x)\cos(x) dx, \qquad I_3 = \int_0^1 x e^x dx.$$

Solution Ex 1. (4.5 pts)

Calculation of I_1 The first integral can be evaluated in a direct way

$$\begin{aligned} I_1 &= \int_1^4 \frac{\sqrt{x}-1}{\sqrt{x}} dx = \int_1^4 \left(\frac{\sqrt{x}}{\sqrt{x}} - \frac{1}{\sqrt{x}} \right) dx \\ &= \int_1^4 (1 - \frac{1}{\sqrt{x}}) dx \\ &= \left[x - 2\sqrt{x} \right]_{x=1}^{x=4} \\ &= 1 \qquad \boxed{1.5pt} \end{aligned}$$

Calculation of I_2 For the second integral, we can use the recognition of the form

$$\int u'(x) u(x) dx = \frac{1}{2} u^2(x) + C.$$

Therefore

$$I_2 = \int_1^{\frac{\pi}{2}} \sin(x)\cos(x) dx = \left[\frac{1}{2} \sin^2(x) \right]_{x=0}^{x=\frac{\pi}{2}} = \frac{1}{2} \qquad \boxed{1.5pt}$$

Calculation of I_3 To evaluate the third integral we can use an integration by parts. For doing so, let us set $u(x) = x$ and $v'(x) = e^x$, so $u'(x) = 1$ and $v(x) = e^x$. Thus, we obtain

$$\begin{aligned} I_3 &= \int_0^1 x e^x dx = [x e^x]_{x=0}^{x=1} - \int_0^1 e^x dx \\ &= (1 \cdot e^1 - 0 \cdot e^0) - [e^x]_{x=0}^{x=1} \\ &= e - (e-1) = 1 \qquad \boxed{1.5pt} \end{aligned}$$

Exercise 2. (8.5 pts)

Consider the matrix M given by

$$M = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}$$

1) Prove that M is invertible.

2) Evaluate M^2 and M^3, and verify that $M^3 - M = 4I_3$.

3) Express M^{-1} in terms of M^2 and I_3, then determinate M^{-1}.

4) Consider the system

$$(\mathrm{S}) \quad \begin{cases} x + 2z = 5 \\ -y + z = 2 \\ x - 2y = 1 \end{cases}$$

   a) Rewrite the system (S) in the matrix formulation.

   b) Prove that the system (S) has only one solution.

   c) Solve the system (S).

Solution Ex 2. (8.5 pts)

1) A matrix is invertible if and only if its determinant is non-zero:

$$\text{If } \det(M) \neq 0 \Rightarrow M \text{ is invertible.}$$

We use cofactor expansion along the first row:

$$\det(M) = 1 \cdot \begin{vmatrix} -1 & 1 \\ -2 & 0 \end{vmatrix} - 0 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + 2 \cdot \begin{vmatrix} 0 & -1 \\ 1 & -2 \end{vmatrix}$$
$$= 1 \cdot 2 + 2 \cdot 1 = 2 + 2 = 4$$

Since $\det(M) = 4 \neq 0$ then M is invertible. $\boxed{1pt}$

2) Evaluate M^2

$$M^2 = M \cdot M \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -4 & 2 \\ 1 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix} \qquad \boxed{0.75pt}$$

Evaluate M^3

$$M^3 = M^2 \cdot M \begin{pmatrix} 3 & -4 & 2 \\ 1 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 1 \\ 1 & -2 & 4 \end{pmatrix} \qquad \boxed{0.75pt}$$

Pass to verify that $M^3 - M = 4I_3$.

$$M^3 - M = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 1 \\ 1 & -2 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 4I_3 \qquad \boxed{0.5pt}$$

3) We have: $M^3 - M = I_3$, We can factor M from the left-hand side: $M \cdot \left(M^2 - I_3\right) = 4I_3$, thus $M \cdot \frac{1}{4}\left(M^2 - I_3\right) = I_3$. According to the definition, we conclude that $M^{-1} = \frac{1}{4}\left(M^2 - I_3\right)$ $\boxed{1pt}$.

$$\begin{aligned} M^{-1} = \frac{1}{4}\left(M^2 - I_3\right) &= \frac{1}{4}\left(\begin{pmatrix} 3 & -4 & 2 \\ 1 & -1 & -1 \\ 1 & 2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) \\ &= \frac{1}{4}\begin{pmatrix} 3-1 & -4 & 2 \\ 1 & -1-1 & -1 \\ 1 & 2 & 0-1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & -4 & 2 \\ 1 & -2 & -1 \\ 1 & 2 & -1 \end{pmatrix} \end{aligned} \qquad \boxed{1pt}$$

4) a) The system (S) can be written as $AX = B$, where

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} \qquad \boxed{1pt}$$

b) Form **(Q1)**, we have $det(M) = det(A) = 4 \neq 0$, so the system (S) has only one solution. $\boxed{1pt}$
c) To solve the system (S) one can apply Crammer's rule or use the inverse of A^{-1} computed previously. $\boxed{1.5pt}$

Crammer rule:

$$x = \frac{\det(A_x)}{det(A)} = \frac{1}{4}\begin{vmatrix} 5 & 0 & 2 \\ 2 & -1 & 1 \\ 1 & -2 & 0 \end{vmatrix} = \frac{4}{4} = 1$$

$$y = \frac{\det(A_y)}{det(A)} = \frac{1}{4}\begin{vmatrix} 1 & 5 & 2 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \frac{0}{4} = 0$$

$$z = \frac{\det(A_z)}{det(A)} = \frac{1}{4}\begin{vmatrix} 1 & 0 & 5 \\ 0 & -1 & 2 \\ 1 & -2 & 1 \end{vmatrix} = \frac{8}{4} = 2$$

Therefore, the solution of the system (S) is: $(x, y, z) = (1, 0, 2)$. **Inverse of matrix method:** The solution of the system $AX = B$ is given explicitly as $X = A^{-1} \cdot B$.

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 & -4 & 2 \\ 1 & -2 & -1 \\ 1 & 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Exercise 3. (7 pts)

1) Consider the following first-order linear differential equation

$$y' - 2xy = (-2x + 1)e^x \tag{E}$$

a) Let $f : \mathbb{R} \to \mathbb{R}$ a function defined as: $f(x) = e^{-x^2+x}$. Determine f' derivative of f.

b) Solve the equation (E).

2) Consider the following second-order linear differential equation

$$y'' - 3y' + 2y = xe^{-x}. \tag{E'}$$

a) Find y_h the solution of the homogeneous differential equation $y_h'' - 3y_h' + 2y_h = 0$.

b) Find y_p a particular solution for (E') of the form $y_p(x) = (ax + b)e^{-x}$ with $a, b \in \mathbb{R}$.

c) Deduce the solution of the differential equation (E').

Solution Ex 3. (7 pts)

1) a) The function f is differentiable on $\mathbb{R}$, using chain rule we obtain

$$f'(x) = (-2x + 1)e^{-x^2+x} \qquad \boxed{0.5pt}.$$

b) (E) is a first-order linear differential equation, thus its solution is given as $y = y_h + y_p$.
Finding y_h homogeneous solution: we known that y_h is the solution of the homogeneous differential equation: $y_h' - 2xy_h = 0$, so $y_h(x) = \lambda e^{x^2}$ with $\lambda \in \mathbb{R}$ $\boxed{1pt}$.
Finding y_p particular solution: we known that $y_p(x) = \lambda(x)e^{x^2}$ is a particular solution for the

equation (E), thus we obtain

$$\begin{aligned} y_p' - 2xy_p = (-2x+1)e^x &\iff \left(\lambda(x)e^{x^2}\right)' - 2x\left(\lambda(x)e^{x^2}\right) = (-2x+1)e^x \\ &\iff \lambda'(x)e^{x^2} + 2x\lambda(x)e^{x^2} - 2x\lambda(x)e^{x^2} = (-2x+1)e^x \\ &\iff \lambda'(x)e^{x^2} = (-2x+1)e^x \\ &\iff \lambda'(x) = (-2x+1)e^{-x^2+x}. \qquad \boxed{1pt} \end{aligned}$$

From **(Q1)-a**, we deduce that $\lambda(x) = e^{-x^2+x}$. Therefore, we obtain

$$y_p(x) = e^{-x^2+x} \times e^{x^2} = e^x. \qquad \boxed{0.5pt}$$

Finding y general solution:

$$y(x) = y_h(x) + y_p(x) = \lambda e^{x^2} + e^x, \quad \lambda \in \mathbb{R}. \qquad \boxed{0.5pt}$$

2) a) First, we write the characteristic equation

$$(E_r): \quad r^2 - 3r + 2 = 0, \boxed{0.5pt}$$

Clearly (E_r) has two simple real solution $r_1 = 1$ and $r_2 = 2$, thus y_h solution of the homogeneous equation is given as:

$$y_h(x) = C_1e^x + C_2e^{2x}, \quad C_1, C_2 \in \mathbb{R}. \qquad \boxed{0.5pt}$$

b) Let us consider a particular solution of the form:

$$y_p(x) = (ax+b)e^{-x}$$

Compute the first and second derivatives:

$$y_p'(x) = (a - ax - b)e^{-x}, \quad y_p''(x) = (-2a + 2ax + 2b)e^{-x}. \qquad \boxed{0.5pt}$$

Substitute into the differential equation:

$$y'' - 3y' + 2y = xe^{-x}$$

$$\begin{aligned} (-2a + 2ax + 2b)e^{-x} - 3(a - ax - b)e^{-x} + 2(ax+b)e^{-x}, \qquad \boxed{0.5pt} \\ = [7ax - 5a + 7b]\, e^{-x} \end{aligned}$$

Set equal to the right-hand side:

$$(7ax - 5a + 7b)e^{-x} = xe^{-x}$$

Match coefficients:

$$7a = 1 \Rightarrow a = \frac{1}{7}, \quad -5a + 7b = 0 \Rightarrow b = \frac{5}{49} \qquad \boxed{0.5pt}$$

Therefore, the particular solution is:

$$y_p(x) = \left(\frac{1}{7}x + \frac{5}{49}\right) e^{-x} \qquad \boxed{0.5pt}$$

c) The general solution of the nonhomogeneous differential equation is the sum of the general solution of the homogeneous equation and a particular solution:

$$y(x) = y_h(x) + y_p(x)$$

From parts (a) and (b), we have:

$$y_h(x) = C_1 e^x + C_2 e^{2x}, \quad y_p(x) = \left(\frac{1}{7}x + \frac{5}{49}\right) e^{-x}$$

Therefore, the general solution is:

$$y(x) = C_1 e^x + C_2 e^{2x} + \left(\frac{1}{7}x + \frac{5}{49}\right) e^{-x}. \qquad \boxed{0.5pt}$$